Dislocation theory as a 3-dimensional translation gauge theory[*]

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We consider the static elastoplastic theory of dislocations in an elastoplastic material. We use a Yang-Mills type Lagrangian (the teleparallel equivalent of Hilbert-Einstein Lagrangian) and some Lagrangians with anisotropic constitutive laws. The translational part of the generalized affine connection is utilized to describe the theory of elastoplasticity in the framework of a translation gauge theory. We obtain a system of Yang-Mills field equations which express the balance of force and moment.

Keywords: Dislocation theory; force stress; moment stress

I. INTRODUCTION

The field theory of defects in crystals has an old history. First, Kondo [1] and Bilby et al. [2] described independently a dislocation theory in the language of differential geometry. They proved the equivalence of dislocation density to Cartan's torsion. Kröner and Seeger [3,4] completed this theory to a non-linear theory of elasticity with dislocations and internal stress. Recent developments of this theory are given in [5,6].

The first step towards a gauge theory of defects had been taken by Turski [7]. He derived the equilibrium equations by means of a variational principle.

Edelen et al. [8,9] developed a theory of dislocations and disclinations as a gauge theory of the Euclidean group $SO(3) \otimes T(3)$ and a pure dislocation theory as a translational or $T(3)$-gauge theory. Unfortunately, they did not distinguish between disclinations in solid and in liquid crystals. Moreover, they did not refer to the gauge theory of the Poincaré group $SO(1,3) \otimes T(4)$ which was very well developed in gravity [10]. Their theory is claimed to describe the dynamics of defects. However, they have ignored the following effects: Dislocations can move in two different modes, called glide and climb. If dislocations climb, they interchange with point defects such as vacancies and/or interstitials. Therefore, it is necessary to include the point defects into a dynamical theory of dislocations. Furthermore, the motion of dislocation is highly dissipative (friction and radiation damping).

Other proposals for gauge theories of dislocations were put forward by Gairola [11,12], Kleinert [13,14], Trzęsowski [15], and, quite recently, by Malyshev [16].

The classical theory of dislocations deals with the incompatibility and the equilibrium conditions (see, e.g., [17]). The drawback of this theory is that a field equation is missing. Therefore, one has to take as ansatz for the dislocation density a singular δ-function. The price to be paid is that the elastic energy is singular. Thus the information about the dislocation core is lost.

The aim of this paper is to develop a $T(3)$-gauge theory of dislocations similar to gravity as $T(4)$-gauge theory. These gauge theories are included in the metric-affine gauge theory (MAG) as given by Hehl et al. [18], where the metric, the coframe, and the connection are independent field variables. We use the tool of MAG in order to derive the gauge theory of dislocations in an elastoplastic material as field theory of elastoplasticity.

We obtain a system of static field equations which determine both, the elastic and the plastic fields. Additionally, the equations contain the equilibrium condition and the Bianchi identity for the torsion (or dislocation density). As shown later, our field equations differ from the field equations proposed by Edelen et al. [8,9]. One reason is that we have used a different and more physically motivated gauge Lagrangian. For compact notation, we will use the calculus of exterior differential forms.

II. GEOMETRICAL FRAMEWORK

In order to gauge the translation group, it is convenient to start with the affine group. Therefore, we consider the affine group in three dimensions $A(3) = GL(3) \otimes T(3)$, where the symbol $\otimes$ denotes the semidirect product.

The mathematical domain of gauge theories is the theory of fibre bundles [19,20]; for rigorous definitions of fibre bundles and connections, see [21,22]. In the case of the affine group, the corresponding fibre bundle is the bundle of affine frames AM. The connection 1-form $\widetilde{\widetilde{\omega}}$ on AM is called *generalized affine connection*. It is an $\mathfrak{a}(3)$-valued 1-form of type ad, where $\mathfrak{a}(3)$ is the Lie algebra of $A(3)$. The corresponding 2-form $\widetilde{\widetilde{\Omega}}$ is the curvature 2-form on AM. It is given by Cartan's structure equation

$$\widetilde{\widetilde{\Omega}} := \widetilde{\widetilde{\mathrm{D}}}\widetilde{\widetilde{\omega}} \equiv \mathrm{d}\widetilde{\widetilde{\omega}} + \widetilde{\widetilde{\omega}} \wedge \widetilde{\widetilde{\omega}}. \tag{1}$$

The symbol $\wedge$ denotes the exterior product, d is the exterior derivative and D the gauge covariant exterior derivative.

Using the subbundle map $\gamma : LM \mapsto AM$, $\widetilde{\widetilde{\omega}}$ can be pulled back to the bundle of linear frames LM,

$$\gamma^* \widetilde{\widetilde{\omega}} = \omega + \phi = \omega^{ab} L_{ab} + \phi^a P_a, \tag{2}$$

where ω is a $\mathfrak{gl}(3)$-valued 1-form on LM, the *linear connection*, and ϕ is an $\mathbb{R}^3$-valued tensorial 1-form on LM, the *translational connection*. Here P_a are the generators of translations $\mathbb{R}^3$ and the generators L_{ab} span the Lie algebra $\mathfrak{gl}(3)$. The pull back of $\widetilde{\widetilde{\Omega}}$ is given by

$$\gamma^* \widetilde{\widetilde{\Omega}} = \Omega + \Phi = \Omega^{ab} L_{ab} + \Phi^a P_a, \tag{3}$$

where

$$\Omega := \mathrm{D}\omega = \mathrm{d}\omega + \omega \wedge \omega \tag{4}$$

is the curvature of the linear connection ω. The *translational curvature* Φ reads

$$\Phi := \mathrm{D}\phi = \mathrm{d}\phi + \omega \wedge \phi. \tag{5}$$

In particular, the translational connection can be decomposed into the soldering 1-form ϑ and an additional 1-form χ,

$$\phi = \vartheta + \chi. \tag{6}$$

In order to ensure the tensorial transformation behaviour of $\vartheta = \vartheta^a P_a$, which is an $\mathbb{R}^3$-valued tensorial 1-form of type id, it is convenient to follow Trautman [23]. We introduce the vector-valued 0-form $\xi \equiv \xi^a P_a$ and identify $\chi = -\mathrm{D}\xi$. With these identifications, the soldering-form turns out to be

$$\vartheta = \widetilde{\widetilde{\mathrm{D}}}\xi \equiv \mathrm{D}\xi + \phi. \tag{7}$$

It transforms as a vector-valued 1-form. With (5) and (7), the torsion 2-form, in the framework of the affine group, reads

$$T \equiv T^a P_a := \mathrm{D}\vartheta = \left(\Phi^a + \Omega^a{}_b \xi^b\right) P_a. \tag{8}$$

Since the translation group is a subgroup of $A(3)$, we obtain, with the choice $\omega \equiv 0$, the translational part of the *generalized affine connection* as

$$\phi = \vartheta - \mathrm{d}\xi. \tag{9}$$

Some remarks on the generalized affine connection in relation to dislocations have been made by Mistura [24].

III. PHYSICAL IDENTIFICATIONS AND CLASSICAL ELASTICITY

The soldering form is given with respect to the translational part of the generalized affine connection ϕ^a by

$$\vartheta^a = \mathrm{d}\xi^a + \phi^a. \tag{10}$$

Eq. (10) is the key formula for the $T(3)$-gauge theory. It is valid in a Weitzenböck space ($\Omega^a{}_b = 0$) under the gauge condition $\omega^a{}_b \equiv 0$. The coupling in eq. (10) between the translational gauge potential ϕ^a and the vector field ξ^a is characteristic for the translation gauge theory as a special case of the metric–affine gauge theory. The reason lies in the nature of the affine group itself.

At first, in agreement with Edelen [8], we identify the translational connection or the translational gauge potential ϕ^a as the gauge potential of the dislocations. This identification is justified by the following ideas. A gauge potential is a 1-form. If we form the corresponding 2-form, there results the gauge field strength. In our case of a Weitzenböck space, in the gauge chosen, the torsion 2-form is equal to the object of anholonomity. Therefore, one obtains

$$T^a = \mathrm{d}\vartheta^a \equiv \mathrm{d}\phi^a = \frac{1}{2} T^a{}_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j, \tag{11}$$

which can be identified with the dislocation density. One recovers the conventional dislocation density tensor $\alpha^a{}_i$ from T^a by means of $\alpha^a{}_i = \frac{1}{2}\epsilon_i{}^{jk} T^a{}_{jk}$. Thus, the dislocation density or the torsion represents the gauge field strength of the translation group. In the framework of $T(3)$-gauge theory ($\omega^a{}_b \equiv 0$), the torsion 2-form satisfies the Bianchi identity

$$\mathrm{d}T^a = 0. \tag{12}$$

The indices $a, b, c, \ldots = 1, 2, 3$ denote the (anholonomic) material or the final coordinates and $i, j, k, \ldots = 1, 2, 3$ the (holonomic) Cartesian coordinates of the reference system (defect-free or ideal reference system)[1]. Now, we identify ξ^a with the diffeomorphisms of the material space into the Euclidean space in which the crystal is embedded. We introduce the displacement field u^a and write ξ^a in terms of it as $\xi^a = \delta^a_i x^i + u^a$. With this identification, the soldering form is specified

$$\vartheta^a = B^a{}_i\, \mathrm{d}x^i = \left(\delta^a_i + \partial_i u^a + \phi^a{}_i\right) \mathrm{d}x^i. \tag{13}$$

The following consideration justifies the last identification: In a material with compatible distortion $\phi^a = 0$, the soldering form is $\vartheta^a = \mathrm{d}\xi^a$. The metric of the final state is the Cauchy-Green strain tensor G which, in our case (teleparallelism), is given by

$$G = \delta_{ab}\, \vartheta^a \otimes \vartheta^b = B^a{}_i B_{aj}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad \delta_{ab} = \mathrm{diag}(+++). \tag{14}$$

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[1] A field of coframes ϑ is *holonomic* if $\mathrm{d}\vartheta = 0$ and *anholonomic* if $\mathrm{d}\vartheta \neq 0$ [25].

The engineering strain tensor is

$$2E = G - 1 = (g_{ij} - \delta_{ij})\mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{15}$$

It is obvious that the soldering 1-form corresponds to the distortion 1-form. If the distortion is compatible, the distortion is given as a deformation gradient of a vector field. This vector field is ξ^a.

In the general case of elastoplasticity, eqs. (10) and (13) describe an incompatible distortion with the distortion 1-form $\boldsymbol{\beta} \equiv \boldsymbol{\vartheta}$. The cause of plasticity are defects and the material gives rise to a specific elastic response. The plastic distortion is given by the dislocation gauge potential $\boldsymbol{\beta}^{\mathrm{pl}} \equiv -\boldsymbol{\phi}$. Finally, the total distortion $\boldsymbol{\beta}^{\mathrm{T}}$ contains elastic and plastic contributions according to

$$\boldsymbol{\beta}^{\mathrm{T}} = \boldsymbol{\beta}^{\mathrm{pl}} + \boldsymbol{\beta}. \tag{16}$$

Since it is compatible, we have

$$\mathrm{d}\boldsymbol{\beta}^{\mathrm{T}} = 0, \qquad \mathrm{d}\boldsymbol{\beta}^{\mathrm{pl}} = -\mathrm{d}\boldsymbol{\beta}. \tag{17}$$

Thus the physical space (crystal) is determined by the two fields $\mathrm{d}\xi^a$ and ϕ^a.

Let us now discuss the concepts of *internal* and *external observers* as introduced by Kröner [26]. The internal observer lives in the crystal and uses ϑ^a as coframe. Consequently, he can detect defects due to $\mathrm{d}\vartheta^a = T^a$. But he misses the information about the holonomic coordinate system and is unable to detect compatible deformations. The external observer lives in the external space in which the crystal is embedded. He has more information available than the internal observer because he knows the holonomic coordinate system and is able to detect compatible deformations. The external observer measures the torsion in his coordinate system as $T^i = B_a{}^i T^a$.

In gravity, the physical meaning of ξ^a is not well-understood [18,27]. It is obvious that we are internal observers in our universe and cannot detect ξ^a or $\mathrm{d}\xi^a$. The internal observer in gravity would use the *Cartan* or *affine connection* $(\vartheta^a, \omega^a{}_b)$ instead of the generalized affine connection. Does, perhaps, this idea also lead to an understanding of ξ^a in gravity?

In order to simplify the formulas, we assume a linear constitutive law (stress-strain relation). But our general considerations are not restricted to this assumption. The elastic (anisotropic) behaviour of the material can be described by the elasticity tensor [28]

$$C = C^{ijkl}\, \partial_i \otimes \partial_j \otimes \partial_k \otimes \partial_l. \tag{18}$$

C is a contravariant tensor of fourth rank with the symmetries

$$C^{ijkl} = C^{jikl} = C^{ijlk} = C^{klij}. \tag{19}$$

The elastic energy (potential energy) contains the constitutive law and is given by

$$W = \frac{1}{2} C : (E \otimes E) = \frac{1}{2} C^{ijkl} E_{ij} E_{kl}, \tag{20}$$

where the symbol : denotes double contraction. Since we consider a static and not a dynamical theory, the elastic Lagrangian is given by means of the potential energy

$$\mathcal{L}_{\mathrm{el}} = -{}^{\star} W = -W \mathrm{d}v_{\mathrm{E}}, \tag{21}$$

where the volume 3-form is defined by

$$\mathrm{d}v_{\mathrm{E}} := \frac{1}{3!} \eta_{abc}\, \vartheta^a \wedge \vartheta^b \wedge \vartheta^c, \tag{22}$$

with $\eta_{abc} := \sqrt{|B|}\epsilon_{abc}$, and ϵ_{abc} as the Levi-Civita symbol. Furthermore, $\eta_a := e_a \rfloor \mathrm{d}v_{\mathrm{E}}$, $\eta_{ab} := e_a \rfloor e_b \rfloor \mathrm{d}v_{\mathrm{E}}$, and $\eta_{abc} := e_a \rfloor e_b \rfloor e_c \rfloor \mathrm{d}v_{\mathrm{E}}$, where $\rfloor$ denotes the interior product with

$$e_a \rfloor \vartheta^b = B_a{}^i B^b{}_i = \delta_a^b, \qquad e_a = B_a{}^i \partial_i. \tag{23}$$

The symbol * denotes the Hodge star operator which, in three dimensions, defines the dual $(3-p)$-form of a given p-form.

IV. GAUGE THEORY OF DISLOCATIONS

Let us now derive the gauge Lagrangian of the translation group $T(3)$ in analogy to gravity [18,29,30]. We make the most general Yang-Mills ansatz which is quadratic in the corresponding field strength $T^a = \mathrm{d}\vartheta^a$,

$$V_{\parallel} = -\frac{1}{2}T^a \wedge H_a. \tag{24}$$

The simplest choice is $H_a = \frac{1}{\ell}\,{}^\star T_a$ which was used by Edelen for a dislocation gauge theory. It is well-known that, in gravity, this Lagrangian does not yield Einstein's theory [31]. Recently, Malyshev [16] discussed the gauge Lagrangian used by Edelen [8,9]. He showed that it does not lead to the correct solutions for edge dislocations within a linear approximation. Accordingly, we will use the most general Lagrangian for an isotropic material with the 1-form

$$H_a = \frac{1}{\ell}\,{}^\star \sum_{I=1}^{3} a_I \,{}^{(I)}T_a. \tag{25}$$

Here ℓ is the coupling constant of the theory. It is obvious that eq. (25) is a constitutive law for an isotropic material. This Lagrangian has to be invariant under local $SO(3)$-transformations in order to obtain the teleparallel version of the Hilbert-Einstein Lagrangian, since the Einstein theory of gravity can be viewed as the gauge theory of the translation group in four dimension. In this framework, the vierbein fields are the translational gauge potentials and the field strength is given in terms of the anholonomity [32].

The gauge Lagrangian (25) was used by Katanaev and Volovich [33] to describe dislocations, but without combining it with an elastic Lagrangian. Thus their field equations do not really describe dislocations in an elastic material.

The condition of local $SO(3)$-invariance yields the following three parameters (see, e.g., [33], [18]),

$$a_1 = -1, \qquad a_2 = 2, \qquad a_3 = \frac{1}{2}\,. \tag{26}$$

Consequently

$$H_a = \frac{1}{\ell}\,{}^\star\big(-{}^{(1)}T_a + 2\,{}^{(2)}T_a + \frac{1}{2}\,{}^{(3)}T_a\big). \tag{27}$$

The three pieces ${}^{(I)}T_a$ are irreducible with respect to $SO(3)$. Thus the torsion or dislocation density reads $T^a = {}^{(1)}T^a + {}^{(2)}T^a + {}^{(3)}T^a$, with the number of independent components $9 = 5 \oplus 3 \oplus 1$, where (for notations see [18])

$$
{}^{(1)}T^a := T^a - {}^{(2)}T^a - {}^{(3)}T^a \qquad \text{(tentor)}, \tag{28}
$$

$$
{}^{(2)}T^a := \frac{1}{2}\vartheta^a \wedge \big(e_b \rfloor T^b\big) \qquad \text{(trator)}, \tag{29}
$$

$$
{}^{(3)}T_a := \frac{1}{3}e_a \rfloor \big(\vartheta^b \wedge T_b\big) \qquad \text{(axitor)}. \tag{30}
$$

In order to obtain these three irreducible pieces, one can use the standard method of Young tableaux. In the field theory of dislocations, the axitor describes three vertical bands of screw dislocations [34].

With the identity (R^{ab}=Riemann-Cartan curvature 2-form, $\widetilde{R}^{ab}$=Riemann-Christoffel curvature 2-form)

$$\widetilde{R}^{ab} \wedge \eta_{ab} = R^{ab} \wedge \eta_{ab} + T^a \wedge {}^\star\big(-{}^{(1)}T_a + 2\,{}^{(2)}T_a + \frac{1}{2}\,{}^{(3)}T_a\big) + 2\mathrm{d}\big(\vartheta^a \wedge {}^\star T_a\big), \tag{31}$$

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in the case of teleparallelism $R^{ab} = 0$ and dropping the surface term, we find the equivalence to the Hilbert-Einstein Lagrangian V_{GR},

$$V_{\|} = V_{\mathrm{GR}} = -\frac{1}{2\ell}\widetilde{R}^{ab} \wedge \eta_{ab} \equiv \frac{1}{2\ell} {}^{\star}\widetilde{R}. \tag{32}$$

Here $\widetilde{R} := e_a \rfloor e_b \rfloor \widetilde{R}^{ab}$. Finally, the total Lagrangian reads

$$\mathcal{L} = V_{\|} + \mathcal{L}_{\mathrm{el}}. \tag{33}$$

Thus, for the screw dislocations in an isotropic material, one needs only one material constant. Accordingly, in this specific case, the ansatz of Edelen et al. [8] coincides with the more general ansatz (25). Moreover, in the framework of Edelen's dislocation theory, Osipov [35] showed that the second order corrections to the stress field of a *screw* dislocation far from the core are in good agreement with the formulas given by Kröner and Seeger [3].

Let us now turn to the anisotropic case. The linear constitutive law for an anisotropic material reads

$$H_a = \frac{1}{2}{}^{\star}\Big(\kappa_{aij}{}^{bkl} T_{bkl}\,\mathrm{d}x^i \wedge \mathrm{d}x^j\Big). \tag{34}$$

A non-linear constitutive law, in analogy to non-linear (Born-Infeld type) electrodynamics, is given by

$$H_a = \frac{1}{2}{}^{\star}\Big(\kappa_{aij}{}^{bkl} T_{bkl}\,\mathrm{d}x^i \wedge \mathrm{d}x^j + \tilde{\kappa}_{aij}{}^{bklcmn} T_{bkl} T_{cmn}\,\mathrm{d}x^i \wedge \mathrm{d}x^j + \dots\Big), \tag{35}$$

where $\kappa_{aij}{}^{bkl}$ and $\tilde{\kappa}_{aij}{}^{bklcmn}$ are the constitutive functions.

The elastic stress tensors, or the corresponding stress forms, are the currents of the elastic field. Therefore, the stress 2-form is defined by

$$\Sigma_a := \frac{\delta\mathcal{L}_{\mathrm{el}}}{\delta\vartheta^a}. \tag{36}$$

In local components, this stress 2-form reads,

$$\Sigma_a = \frac{1}{2}\Big(\sigma^{kl} B_{ak}\eta_{lmn} - \frac{1}{2}\sigma^{kl} E_{kl}\eta_{amn}\Big)\mathrm{d}x^m \wedge \mathrm{d}x^n, \tag{37}$$

with $\sigma^{kl} = C^{ijkl} E_{ij}$. The first Piola-Kirchhoff stress tensor is included in the 1-form dual to the stress 2-form

$${}^{\star}\Sigma_a = t_{al}\,\mathrm{d}x^l = \Big(\sigma^k{}_l B_{ak} - \frac{1}{2}\sigma^{ij} E_{ij} B_{al}\Big)\mathrm{d}x^l, \tag{38}$$

where t_{al} is the first Piola-Kirchhoff stress tensor. The second Piola-Kirchhoff stress tensor, derived from the first Piola-Kirchhoff stress tensor, is given by

$$t_{ac} = B_c{}^l\, t_{al} = \sigma_{ac} - \frac{1}{2}\delta_{ac}\,\sigma^{ij} E_{ij}. \tag{39}$$

Here t_{ac} corresponds to the Maxwell tensor of elasticity for the incompatible case. The symmetric Cauchy stress tensor arises as

$$t_{ij} = B^a{}_i\, t_{aj} = \sigma_{ij} - \frac{1}{2}g_{ij}\,\sigma^{kl} E_{kl}. \tag{40}$$

If dislocations are present, the stress forms are the elastic responses of the body to the dislocations.

The gauge field momentum 1-form (nowadays called excitation) is defined by

$$H_a := -\frac{\partial V_{\|}}{\partial T^a}, \tag{41}$$

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i.e., it is the specific "response" quantity of the gauge Lagrangian $V_\|$ (and not of the elastic Lagrangian $\mathcal{L}_{\text{el}}$) to T^a. It has the dimension of a moment stress. Thus, we interpret H_a as a moment or couple stress originating in the dislocation core. The moment stress 1-form, in local components for an anisotropic linear material, reads

$$H_a = H_{an} \mathrm{d}x^n = \kappa_a{}^{ijbkl} \, T_{bkl} \, \eta_{ijn} \, \mathrm{d}x^n. \tag{42}$$

On the other hand, the translation gauge current or the stress 2-form of the gauge field,

$$h_a := \frac{\partial V_\|}{\partial \vartheta^a} = e_a \rfloor V_\| + \left(e_a \rfloor T^b \right) \wedge H_b, \tag{43}$$

explicitly reads

$$h_a = \frac{1}{2} \left[\left(e_a \rfloor T^b \right) \wedge H_b - T^b \left(e_a \rfloor H_b \right) \right]. \tag{44}$$

Whereas Σ_a is the stress 2-form of the elastic material, h_a is the response stress 2-form of the dislocations or the plastic stress 2-form. Accordingly, h_a is the internal stress caused by dislocations.

The field equations are derived by varying the total Lagrangian with respect to the elastic field ξ^a and the gauge potential ϕ^a:

$$\frac{\delta \mathcal{L}}{\delta \xi^a} \equiv \mathrm{d}\Sigma_a + \mathrm{d}h_a = 0, \tag{45}$$

$$\frac{\delta \mathcal{L}}{\delta \phi^a} \equiv \mathrm{d}H_a - h_a = \Sigma_a. \tag{46}$$

Eq. (46) is the Yang-Mills type gauge field equation in the theory of elastoplasticity. Both stress 2-forms, Σ_a and h_a, are sources in eq. (46). Eq. (45) is a consequence of (46) and the Poincaré lemma $\mathrm{dd} = 0$. The Euler-Lagrange equations can be interpreted as equilibrium equations. Eq. (45) describes the force and eq. (46) the moment equilibrium. The fields to be determined from the field equations are ξ^a or u^a and ϕ^a. Due to the nonlinear geometrical character of elastoplasticity, the field equations are represented by a coupled system of nonlinear partial differential equations. In the framework of MAG, eq. (45) is the matter field and eq. (46) the first gauge field equation (see [18]).

The force density 3-forms are defined as follows: The Peach-Koehler force [36] (elastic force acting on a dislocation), which is analogously defined as the Lorentz force in Maxwell's theory, reads

$$f_a^{\text{el}} := \mathrm{d}\Sigma_a = \left(e_a \rfloor T^b \right) \wedge \Sigma_b. \tag{47}$$

The response force to dislocations is

$$f_a^{\text{g}} := \mathrm{d}h_a = \left(e_a \rfloor T^b \right) \wedge h_b. \tag{48}$$

Eq. (45) describes the force equilibrium between the Peach-Koehler force and the dislocation-response force

$$0 = f_a = f_a^{\text{g}} + f_a^{\text{el}} = \mathrm{d}h_a + \mathrm{d}\Sigma_a, \tag{49}$$

or is, equivalently, interpreted as the equilibrium condition of an elastic body containing dislocations. Both these forces are configurational forces. It is to be emphasized that (47) and (48), in the framework of field theory, are the first Noether identities for the elastic and the gauge fields.

For vanishing dislocation density ($\phi^a = 0$ and $T^a = 0$), the field equations reduce to the equilibrium condition of the classical elasticity theory under zero external forces

$$\mathrm{d}\Sigma_a = 0 \quad \Rightarrow \quad \partial^i t_{ai} = 0. \tag{50}$$

V. RELATIONS TO OTHER DESCRIPTIONS OF DISLOCATION THEORY

In an alternative description of teleparallelism one replaces $T^a \equiv \mathrm{d}\vartheta^a$ by the Levi-Civita connection $\widetilde{\omega}^a{}_b$ of the metric (Cauchy-Green tensor) $G = \delta_{ab}\vartheta^a \otimes \vartheta^b$. One applies Cartan's first structure equation

$$\mathrm{d}\vartheta^a = -\widetilde{\omega}^a{}_b \wedge \vartheta^b, \tag{51}$$

which yields

$$\widetilde{\omega}_{ab} = \frac{1}{2}\big(e_a \rfloor T_b - e_b \rfloor T_a - (e_a \rfloor e_b \rfloor T_c) \wedge \vartheta^c\big). \tag{52}$$

The corresponding Riemannian curvature 2-form reads

$$\widetilde{R}_{ab} = \mathrm{d}\widetilde{\omega}_{ab} + \widetilde{\omega}_{ac} \wedge \widetilde{\omega}^c{}_b. \tag{53}$$

Eventually, we get the corresponding field equation by substituting eq. (53) into the Hilbert-Einstein Lagrangian V_{GR} of eq. (32). After variation, one recovers the Einstein type field equation

$$\frac{1}{2}\eta_{abc}\widetilde{R}^{bc} = \ell\,\Sigma_a \tag{54}$$

used by Malyshev [16] for dislocation theory. A disadvantage of this description is that the original structures are blurred and that the field equation does not have the Yang-Mills form.

Now, we can completely translate our formulas of the gauge theoretical description of dislocation theory into the coordinate system of the external observer (holonomic coordinates). The linear connection of the external observer is defined as pure gauge,

$$\omega^i{}_j = B_a{}^i \mathrm{d}B^a{}_j = B_a{}^i\, \partial_k B^a{}_j\, \mathrm{d}x^k. \tag{55}$$

In holonomic coordinates, the torsion is given by

$$T^i = B_a{}^i\, \partial_{[j} B^a{}_{k]}\, \mathrm{d}x^j \wedge \mathrm{d}x^k = \omega^i{}_k \wedge \mathrm{d}x^k. \tag{56}$$

Cartan's torsion tensor $T^i{}_{jk}$ is the antisymmetric part[2] of the components of the connection $\omega^i{}_{jk}$, namely $\frac{1}{2}T^i{}_{jk} \equiv \omega^i{}_{[jk]}$. By means of the torsion (56) and the condition of vanishing nonmetricity,

$$\mathrm{D}g_{ij} \equiv \mathrm{d}g_{ij} - \omega^k{}_i g_{jk} - \omega^k{}_j g_{ik} = 0. \tag{57}$$

The connection $\omega^i{}_j$ can be decomposed into the Levi-Civita connection $\widetilde{\omega}^i{}_j$ and the contortion $\tau^i{}_j$, which is a tensorial 1-form of type ad,

$$\omega^i{}_j = \widetilde{\omega}^i{}_j - \tau^i{}_j. \tag{58}$$

The local components of the Levi-Civita connection and of the contortion read, respectively,

$$\widetilde{\omega}^i{}_j = \frac{1}{2}g^{il}\big(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}\big)\mathrm{d}x^k \tag{59}$$

and

$$\tau^i{}_j = \frac{1}{2}\left(T^i{}_{jk} + T_{jk}{}^i - T_k{}^i{}_j\right)\mathrm{d}x^k. \tag{60}$$

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[2]We will be using the notation $A_{[ij]} \equiv \frac{1}{2}(A_{ij} - A_{ji})$.

We can resolve (60) with respect to the torsion,

$$\tau^i{}_j \wedge \mathrm{d}x^j = -\frac{1}{2}T^i{}_{jk}\mathrm{d}x^j \wedge \mathrm{d}x^k = -T^i. \tag{61}$$

The forces are

$$f_i^{\mathrm{el}} = \mathrm{D}\Sigma_i \equiv \mathrm{d}\Sigma_i + \omega_i{}^j \wedge \Sigma_j = T^j{}_i \wedge \Sigma_j \tag{62}$$

and

$$f_i^{\mathrm{g}} = \mathrm{D}h_i = T^j{}_i \wedge h_j. \tag{63}$$

In holonomic coordinates, the field equations read

$$\mathrm{D}\Sigma_i + \mathrm{D}h_i = 0 \qquad \text{(force equilibrium)}, \tag{64}$$
$$\mathrm{D}H_i = h_i + \Sigma_i \qquad \text{(moment equilibrium)}. \tag{65}$$

Using the condition of teleparallelism $R^{ij} = 0$, in linear approximation we obtain Kröner's incompatibility equation

$$\mathrm{inc}\, E \equiv \nabla \times E \times \nabla = \eta. \tag{66}$$

The symmetric second rank tensor η is called incompatibility tensor; it encompasses the dislocation density. With $H_i = 0$, we recover a dislocation theory without moment stress which is in agreement with the dislocation theory given by Kröner and Seeger [3,4].

VI. CONCLUSION

We have proposed a dislocation gauge theory in an elastoplastic material. The basic equations (12), (45), (46) and (49) of dislocation gauge theory can be summarized in axiomatic way in analogy to the Maxwell theory (for an axiomatic formulation of Maxwell's theory, see, e.g., [37,38]). As soon as the Lagrangian is specified, one can find the basic laws. As gauge Lagrangians we use the teleparallel one, which is equivalent to the Hilbert-Einstein Lagrangian, and some Lagrangians for anisotropic constitutive laws. For linear constitutive laws, the total Lagrangian has the following symbolic form

$$\mathcal{L} \sim (\text{strain})^2 + (\text{dislocation density})^2. \tag{67}$$

The *first law* expresses the force equilibrium

$$\mathrm{d}\Sigma_a^{\mathrm{T}} = 0 \qquad \text{with} \qquad \Sigma_a^{\mathrm{T}} = \Sigma_a + h_a. \tag{68}$$

A consequence of eq. (68), in analogy to the inhomogeneous Maxwell equation, is the inhomogeneous Yang-Mills equation

$$\mathrm{d}H_a = \Sigma_a^{\mathrm{T}} \qquad \text{(moment equilibrium)}. \tag{69}$$

The definition of the elastoplastic force is the *second law*

$$f_a = (e_a \rfloor T^b) \wedge \Sigma_b^{\mathrm{T}}. \tag{70}$$

The conservation law of dislocation density (homogeneous Yang-Mills equation or Bianchi identity) is the *third law*

$$\mathrm{d}T^a = 0. \tag{71}$$

The constitutive laws $\sigma \sim E$ and $H \sim T$, which are the physical input from experimental data, are the *fourth law*.

We have compared our proposal with Kröner's geometric theory of dislocation. Our gauge theoretical formulation of dislocation theory includes Kröner's basic equations and is thus a straightforward description of dislocation theory with moment stress as given previously in [39,40].

Kröner [41], Stojanović [42], and, later, Kleinert [13] have introduced the concept of a double gauge theory of dislocations, that is the stress tensor can be considered as an Einstein tensor of a formal stress space with torsion. From the geometrical and field theoretical point of view, there is no need to interpret the stress as an Einstein tensor. The generalization of the stress tensor is the energy-momentum tensor. The stress tensor is nothing else but the source of the Einstein tensor, and the equilibrium condition

$$\mathrm{div}\,\sigma = 0 \tag{72}$$

is the first Noether identity. Therefore, we did not make use of the concepts of formal stress space and strain space.

Now some remarks are in place with respect to the moment stress caused by disclinations. The generalization of (material) moment stress is the hypermomentum $\Delta_{ab} = \delta\mathcal{L}_{\mathrm{el}}/\delta\omega^{ab}$ which contains the spin current [43]. The hypermomentum is the source of the $GL(n,\mathbb{R})$-gauge field in MAG and it does not appear in a pure translation gauge theory. Therefore, there are no degrees of freedom for describing spin-disclination in a $T(3)$-gauge theory, in agreement with Kröner [44]. Spin-disclinations are defects in materials with microstructure such as liquid crystals or magnetic spin systems, where spin-moment stress occurs. A gauge theory of materials with microstructure was given by Lagoudas [45].

The goal of this paper was the formulation of a dislocation theory as a gauge theory in analogy to gravity. We combined the physical ideas of Kröner's geometrical theory with the framework of MAG. We found a dislocation theory with moment stress H_a and two kinds of force stresses Σ_a and h_a. In this picture, the dislocation is a source of nontrivial torsion in a Weitzenböck space or of Riemannian curvature in a Riemann space, respectively. The elastic material plays the role of a kind of an (an)isotropic "ether" in analogy to the vacuum in gravity theory. The metric (Cauchy-Green strain tensor) g_{ij} is an effective quantity determined by ϕ^a and $\mathrm{d}\xi^a$. However, it is not a gauge potential.

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